Predicting relative permeability from water retention: A direct approach based on fractal geometry

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[1] Commonly, a soil's relative permeability curve is predicted from its measured water retention curve by fitting equations that share parameters between the two curves (e.g., Brooks/Corey-Mualem and van Genuchten-Mualem). We present a new approach to predict relative permeability by direct application of measured soil water retention data without any fitting procedures. The new relative permeability model, derived from a probabilistic fractal approach, appears in series form as a function of suction and the incremental change in water content. This discrete approach describes the drained pore space and permeability at different suctions incorporating the effects of both pore size distribution and connectivity among water-filled pores. We compared the new model performance predicting relative permeability to that of the van Genuchten-Mualem (VG-M) model for 35 paired data sets from the Unsaturated Soil hydraulic Database (UNSODA) and five other previously published data sets. At the 5% level of significance, the new method predicts relative permeabilities from the UNSODA database significantly better (mean logarithmic root-mean-square error, LRMSE = 0.813) than the VG-M model (LRMSE = 1.555). Each prediction of relative permeability from the five other previously published data sets was also significantly better.

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1. Introduction

[2] Knowledge of a soil's unsaturated hydraulic conductivity or relative permeability is critical for describing the flow of fluids and solutes in the vadose zone. Solution of the partial differential equations governing flow under variably saturated conditions requires appropriate constitutive relationships among permeability, saturation, and capillary pressure.

[3] Because measurement of relative permeability is difficult, attempts to predict this function from measurements of water retention have proliferated. Most attempts rely on, more or less, an empirical or fractal description of the drainage process combined with *Burdine*'s [1953] or *Mualem*'s [1976] integral equations to develop a relative permeability function that shares parameters with the water retention function [e.g., *Brooks and Corey*, 1964; *van Genuchten*, 1980; *Tyler and Wheatcraft*, 1990; *Fuentes et al.*, 1996; *Xu and Dong*, 2004]. Sharing parameters between the water retention and relative permeability functions allows the prediction of one function if the other is known but only through the use of and the assumptions that go along with the Burdine and Mualem models. Oversimplified representation of pore space geometry as a bundle of capillary tubes may result in discrepancies when comparing predictions from these models with the results of experimental data [*Fischer and Celia*, 1999; *Tuller and Or*, 2002].

[4] Pore network modeling is an alternative approach to predict relative permeability from measured water retention data [*Fischer and Celia*, 1999; *Vogel and Roth*, 2001; *Metzger et al.*, 2007]. It involves optimization of bond and site size distributions in artificially generated lattices. However, nonunique solutions are easily obtained since various configurations of the pore size distribution and interconnectivity can match those predicted by the measured water retention data [*Vogel and Roth*, 2001].

[5] Fractals are iterative geometrical models for describing irregular and fragmented systems. Fractal geometry has been widely applied to derive physically based expressions for soil hydraulic functions [e.g., *Giménez et al.*, 1997; *Bird et al.*, 2000; *Wang et al.*, 2005]. Most fractal models, however, do not include an explicit description of incomplete pore connectivity, which can result in partial drainage of pores as suction is increased. Thus, estimates of physically based parameters, such as the mass fractal dimension, obtained by fitting these models to experimental water retention data, may not be accurate. Because of incomplete drainage the resulting parameters might better be described as apparent values.

[6] During drainage of a random porous medium, both the pore size distribution and the connectivity of pores determine the drained pore volume as function of suction. *Cihan et al.* [2007] presented a probabilistic fractal approach to describe a drained pore space that explicitly incorporates the effect of connectivity among pores with different sizes

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and allows continuing drainage of pores at different suctions. In this study, we present a discrete version of their water retention model and use it to derive a new expression for the relative permeability function. This approach allows prediction of the relative permeability directly from measured water retention data. Model fitting or parameters are not required. We tested the performance of the new relative permeability expression using soil hydraulic data from the Unsaturated Soil Hydraulic Database (UNSODA) [*Leij et al.*, 1996] and other highly cited data sets collected from the literature.

2. Theory

2.1. Water Retention Function

[7] *Cihan et al.* [2007] introduced a framework to quantitatively describe incomplete drainage and the water retention function during drying of a random mass prefractal porous medium. Their conceptual model assumes that as drying occurs, not all pores of a given size drain at the appropriate suction because of incomplete pore connectivity. The number of solids, N_s, of length r in the mass prefractal porous medium is given by

$$N_s(r) = \left(b^i\right)^D, \quad (1)$$

where i is the iteration level, b is the scale factor defined as the ratio of solid sizes at two successive iteration levels (r_i/r_{i+1}), and D is the mass fractal dimension defined as $\log\,[N_s(r)/N_s(br)]/\log\, b$. The number of pores, N_p, of length r can be expressed as

$$N_p(r) = \left(b^E - b^D\right)b^{(i-1)D}, \quad (2)$$

where E is the Euclidean dimension. Depending upon the lacunarity of the prefractal porous medium, pores of length r_0/b (where r_0 is the characteristic length of the porous medium) that do not drain at the appropriate suction may remain full or drain into pores of length $r_0/b^2, r_0/b^3 \ldots, r_0/b^i$ as the suction, $h \rightarrow \infty$. *Cihan et al.* [2007] proposed an approach to model this complex drainage process. Figure 1 shows a conceptual representation of their approach for a 2-D random mass prefractal porous medium, or randomized Sierpinski carpet, with a unit length, $b = 4$, $D = \log10/\log4 \cong 1.660$ and n (last iteration level of the porous medium) = 2. An initially saturated porous medium ($j = 0$, Figure 1a) begins to drain by applying suction. A no-flow boundary is present on the left- and right- sides, and all drained pores are assumed to retain a water film of negligible volume. Six large pores with a length of 1/4 are present. At the first drainage step ($j = 1$), five of the six large pores that are connected from top to bottom, drain as shown in white in Figure 1b. The large nondraining pore at the bottom left side is connected with smaller pores of length of 1/16 and only drains at the next suction level ($j = 2$, Figure 1c). Nine of the sixty 1/16-sized pores are disconnected and remain water filled at the end of this drying cycle (Figure 1c).

[8] Generalization of the above procedures to a porous medium with arbitrary fractal dimension and scale factor can be accomplished as follows. At drainage level 1, which corresponds to the air entry value or minimum capillary pressure, a P_1 fraction of the $r_1(= r_0/b)$ pores drains. This drained fraction can be expressed as $P_1 N_p^{(1)}/b^E$, where $N_p^{(1)}/b^E$ is the

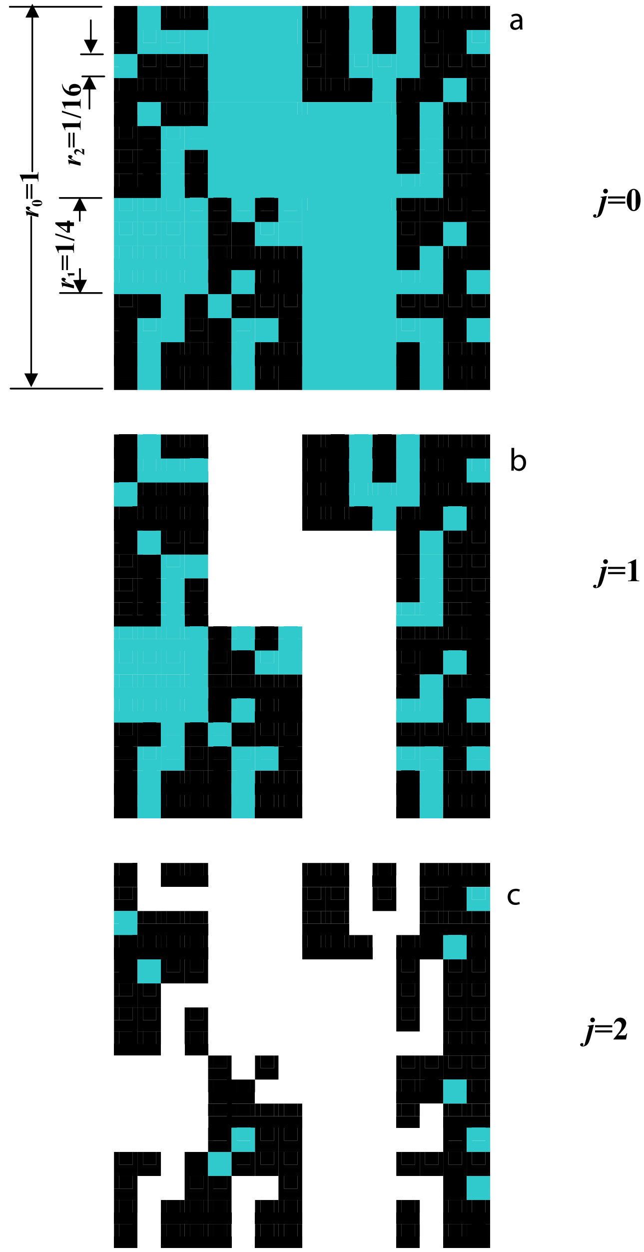

Figure 1. Idealized realization of a 2-D drying random prefractal porous medium with $b = 4$, the last iteration level, $n = 2$ and $D \cong 1.660$ (black is solid, white is air-filled pore, blue is water-filled pore, and $0 \leq j \leq n$ is the drainage level.

proportion of the largest pores within the entire volume. At drainage level 2, a P_2 fraction of the $r_2(= r_0/b^2)$ pores and water-filled r_1 pores drains. The total volume fraction of water draining at $j = 2$ can be expressed by $P_2[N_p^{(2)}/b^{2E} + (1 - P_1)N_p^{(1)}/b^E]$. Cumulatively, after two drainage steps the volumetric fraction of water remaining in the r_0/b- and r_0/b^2-sized pores is given by

$$\left[\frac{N_p^{(2)}}{b^{2E}} + (1 - P_1)\frac{N_p^{(1)}}{b^E}\right] - P_2\left[\frac{N_p^{(2)}}{b^{2E}} + (1 - P_1)\frac{N_p^{(1)}}{b^E}\right]$$
$$= (1 - P_2)\frac{N_p^{(2)}}{b^{2E}} + (1 - P_2)(1 - P_1)\frac{N_p^{(1)}}{b^E}.$$

The proportion of water draining from the connected pores between any two successive drainage levels can be generalized as

$$
\begin{aligned}
j=1 \qquad & \theta_0-\theta_1=P_1\frac{N_p^{(1)}}{b^E},\\
j=2 \qquad & \theta_1-\theta_2=P_2\left[\frac{N_p^{(2)}}{b^{2E}}+(1-P_1)\frac{N_p^{(1)}}{b^E}\right],\\
j=3 \qquad & \theta_2-\theta_3=P_3\left[\frac{N_p^{(3)}}{b^{3E}}+(1-P_2)\frac{N_p^{(2)}}{b^{2E}}\right.\\
& \qquad\qquad \left.+(1-P_2)(1-P_1)\frac{N_p^{(1)}}{b^E}\right],\\
\vdots \qquad & \vdots\\
j \qquad & \theta_{j-1}-\theta_j=P_j\left[\frac{N_p^{(j)}}{b^{jE}}+\left(1-P_{j-1}\right)\frac{N_p^{(j-1)}}{b^{(j-1)E}}\right.\\
& \qquad \left.+\cdots+\left(1-P_{j-1}\right)\left(1-P_{j-2}\right)\ldots(1-P_1)\frac{N_p^{(1)}}{b^E}\right], \qquad (3)
\end{aligned}
$$

where θ_1, θ_2, θ_3, $\ldots\infty$ are the volumetric water contents corresponding to the suction levels j = 1,2,3,. . .∞, θ_0 is the saturated water content, and P_j is the probability of drainage of the remaining pore volume at any j or any corresponding suction. Summation of all terms in equation (3) gives the cumulative drained water content, θ_d, at any j or any corresponding capillary pressure, which is expressed in symbolic form as

$$
\theta_d(j)=\sum_{i=1}^{j}\sum_{m=i}^{j}\left\{\frac{N_p^{(i)}}{b^{iE}}\left[P_m\prod_{k=i}^{m-1}(1-P_k)\right]\right\}, \qquad (4)
$$

where $\Pi_a^b\,()=1$, $\Sigma_a^b\,()=0$ for $a>b$. Then, if the water content at the jth suction level is defined by

$$
\theta_j=\theta_0-\theta_d(j), \qquad (5)
$$

the saturated water content or porosity can be obtained by summing the volume fractions of all pore sizes as $\theta_0=\Sigma_{i=1}^{n}Np^{(i)}/b^{iE}$. The drained water content at the first level of drainage j = 1 is given by $\theta_d(1)=P_1N_p^{(1)}/b^E$. Subtracting $\theta_d(1)$ from θ_0, we obtain the water content at the first level of drainage, i.e., $\theta_1=\theta_0-\theta_d(1)=(1-P_1)N_p^{(1)}/b^E+\Sigma_{i=2}^{n}N_p^{(i)}/b^{iE}$. The general symbolic expression for the water content at any drainage level j can be expressed as

$$
\theta_j=\sum_{i=1}^{j}\left[\frac{N_p^{(i)}}{b^{iE}}\prod_{k=i}^{j}(1-P_k)\right]+\sum_{i=j+1}^{n}\frac{N_p^{(i)}}{b^{iE}}. \qquad (6)
$$

Invoking the Young-Laplace expression [*de Gennes et al.*, 2004], the term b^i in equation (6) can be replaced with the normalized capillary pressure, $h/h_{\min}$. The P values contain information about the connectivity of the pore system and are independent of any assumed fractal morphology for the porous medium. Rearranging equation (3) for P yields [*Cihan et al.*, 2007]

$$
P_j=\frac{\theta_{j-1}-\theta_j}{\sum_{i=1}^{j}\frac{N_p^{(i)}}{b^{iE}}\prod_{k=i}^{j-1}(1-P_k)}. \qquad (7)
$$

The individual P_1, P_2, . . ., P values in equation (7) can be estimated inversely from the water retention curve if b and D values are known a priori for a fractal porous medium [*Cihan et al.*, 2007].

2.2. Relative Permeability: Probabilistic Capillary Connectivity Model

[9] Neglecting inertial effects, the mean velocity of a fluid, u, in a narrow tube of radius, r_t, is given by Poiseuille's equation, i.e.,

$$
u=-\frac{Cr_t^2\rho g}{\mu}\frac{dh}{dl}, \qquad (8)
$$

where C is a shape factor, μ is the dynamic viscosity, ρ is the density, g is the gravitational acceleration, and dh/dl is the potential gradient driving the fluid flow in the tube. If the porous medium is considered to be made up of channels of different sizes, Poiseuille's equation approaches Darcy's law, which expresses the mean velocity of a fluid in a porous medium and can be written as

$$
q=-\frac{k\rho g}{\mu}\frac{dh}{dl}, \qquad (9)
$$

where k is equivalent to $\langle Cr_t^2\rangle$, an averaged quantity for the porous medium. The shape factor C, changes depending on the geometry of the pore.

[10] *Cihan et al.* [2009] first proposed the "probabilistic capillary connectivity model" (PCC) to describe the permeability of saturated porous media. Their approach separates the system into multiple connected flow paths or networks. We employ this same methodology to formulate the relative permeability function. Consider an initially saturated network consisting of only the largest pores of size r_0/b connected from one end to the other in the direction of flow. Following Poiseuille's equation, the mean velocity of water following such a pathway is proportional to r_0^2/b^2. At level 1 of the drainage process, the permeability decreases when the connected proportion of the largest pores of size r_0/b drain. The probability for the existence of such draining pores is provided in equation (3) (i.e., $P_1N_p^{(1)}/b^E$). The remaining proportion of water in pores of size r_0/b is given by $(1-P_1)N_p^{(1)}/b^E$; these pores fail to drain because they are unconnected or connected with smaller pores of size r_0/b^2. There might also be a network formed only of r_0/b^2-sized pores. The probability for the existence of a network containing r_0/b^2-sized pores or r_0/b- and r_0/b^2-sized pores is written as $P_2(N_p^{(2)}/b^{2E}+(1-P_1)N_p^{(1)}/b^E)$, where P_2, as defined previously, represents the connected proportion of water-filled pores in networks formed by r_0/b^2- or larger-sized pores, which will drain at level 2. Since flow rate is controlled by

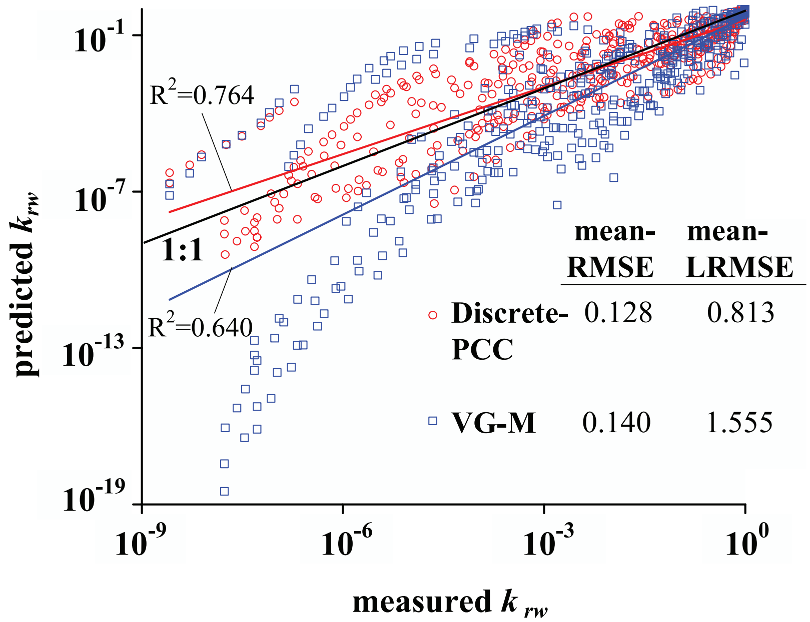

Figure 2. Discrete PCC and VG-M relative permeability predictions for all 40 data sets.

the smallest pores within a serial network of pores, the mean velocity of water flowing along a pathway consisting of r_0/b- and r_0/b^2-sized pores is assumed to be proportional to the cross-sectional area of the smaller pores (i.e., r_0/b^2).

[11] In generalized form, the pore areas controlling flow in the different flow paths, multiplied by their associated probabilities, can be written as

$$\begin{aligned}
j&=1 \qquad P_1\frac{N_p^{(1)}}{b^E}\frac{r_0^2}{b^2},\\
j&=2 \qquad P_2\left[\frac{N_p^{(2)}}{b^{2E}}+(1-P_1)\frac{N_p^{(1)}}{b^E}\right]\frac{r_0^2}{b^4},\\
j&=3 \qquad P_3\left[\frac{N_p^{(3)}}{b^{3E}}+(1-P_2)\frac{N_p^{(2)}}{b^{2E}}+(1-P_2)(1-P_1)\frac{N_p^{(1)}}{b^E}\right]\frac{r_0^2}{b^6},\\
&\vdots \qquad \vdots\\
j&=n \qquad P_n\left[\frac{N_p^{(n)}}{b^{nE}}+(1-P_{n-1})\frac{N_p^{(n-1)}}{b^{(n-1)E}}\right.\\
&\qquad \left.+\cdots+(1-P_{n-1})(1-P_{n-2})\ldots(1-P_1)\frac{N_p^{(1)}}{b^E}\right]\frac{r_0^2}{b^{2n}}.
\end{aligned} \tag{10}$$

The intrinsic permeability is defined by the expected value, $\langle Cr_t^2\rangle$ i.e., the summation of all the terms above leading to [*Cihan et al.*, 2009]

$$k = Cr_0^2\sum_{i=1}^{n}\sum_{m=i}^{n}\left\{\frac{N_p^{(i)}}{b^{iE}}\left[\frac{P_m}{b^{2m}}\prod_{k=i}^{m-1}(1-P_k)\right]\right\}, \tag{11}$$

where n is the last iteration level of the fractal porous medium and the pore shape factor C is assumed to be constant for all pores. Assuming that drainage occurs from the largest pores to the smallest pores sequentially, the permeability of the draining random mass prefractal porous medium can be expressed as

$$k_w(j) = k - Cr_0^2\sum_{i=1}^{j}\sum_{m=i}^{j}\left\{\frac{N_p^{(i)}}{b^{iE}}\left[\frac{P_m}{b^{2m}}\prod_{k=i}^{m-1}(1-P_k)\right]\right\}, \tag{12}$$

where $1 \le j \le n$ is the jth drainage step or suction level. Comparing equation (10) with equation (3), we can rewrite equation (12) as

$$k_w(j) = k - C\sum_{i=1}^{j}\Delta\theta_i\frac{r_0^2}{b^{2i}}, \tag{13}$$

where $\Delta\theta_i = \theta_{i-1} - \theta_i$. Since relative permeability is defined by $k_{rw} = k_w/k$, equation (13) can be modified to

$$k_{rw}(j) = 1 - \frac{\sum_{i=1}^{j}\Delta\theta_i/b^{2i}}{\sum_{i=1}^{n}\Delta\theta_i/b^{2i}}. \tag{14}$$

When j is equal to n, k_{rw} is equal to zero. By applying the Young-Laplace expression, $b^i = h/h_{\min}$ [*de Gennes et al.*, 2004], equation (14) can be expressed in terms of the suction h, giving

$$k_{rw}(j) = 1 - \frac{\sum_{i=1}^{j}\Delta\theta_i/h_i^2}{\sum_{i=1}^{n}\Delta\theta_i/h_i^2}. \tag{15}$$

Equation (15) enables the relative permeability to be predicted directly from the measured data pairs in a water

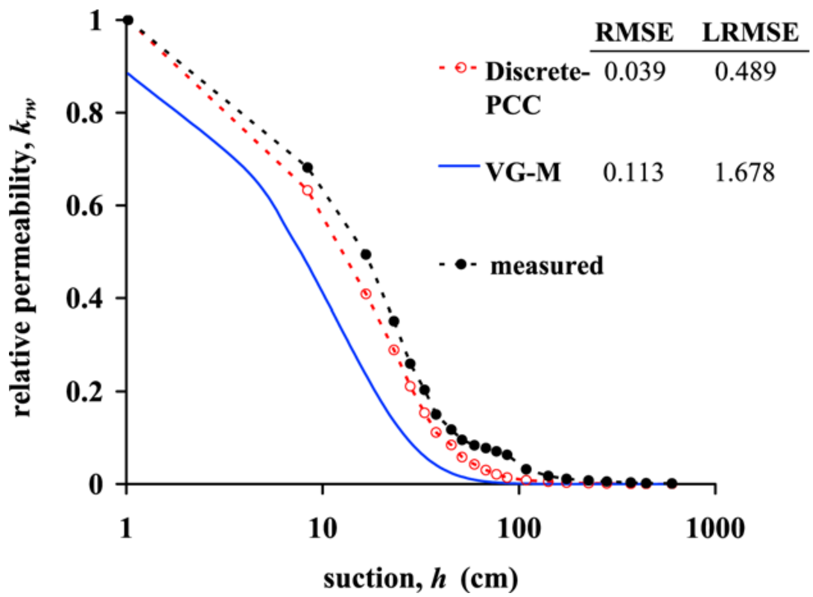

Figure 3. Discrete PCC and VG-M relative permeability predictions for Yolo light clay.

retention curve without the need for any model fitting. Because of the discrete nature of the PCC model it may be quite efficient for numerically solving inverse unsaturated flow and transport problems. However, interpolation between calculated k_{rw} values will require some form of prediction between measured points in the water retention curve.

[12] The current model presented in this paper applies to monotonic drainage only. Thus, water retention data that are to be used to compute relative permeability should represent a monotonically decreasing function. In general, during water retention tests, as suction is incrementally increased, corresponding water content values are recorded. The number of paired water content–suction points obtained during a drainage test may vary according to soil texture, experimental method and equipment used, time available for data collection, etc. Errors associated with discretization can be minimized by choosing small increments.

[13] Natural porous media are often assumed to be composed of continuously distributed pore sizes. For this case, equation (13) can be approximated as

$$k_w = k - C\int_{r}^{r_{\max}} \theta'(s)s^2 ds, \tag{16}$$

where s is a dummy variable for r and the relative permeability is given by

$$k_{rw} = 1 - \frac{\int_{h_{\min}/h}^{1} \theta'(r^*)r^{*2}dr^*}{\int_{h_{\min}/h_{\max}}^{1} \theta'(r^*)r^{*2}dr^*}; \qquad r^* = \frac{r}{r_{\max}} = \frac{h_{\min}}{h}, \tag{17}$$

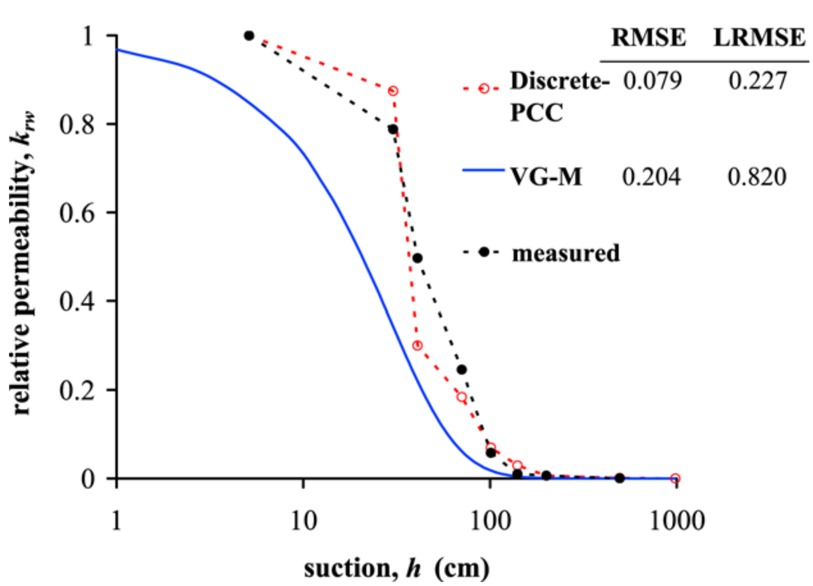

Figure 4. Discrete PCC and VG-M relative permeability predictions for Guelph loam.

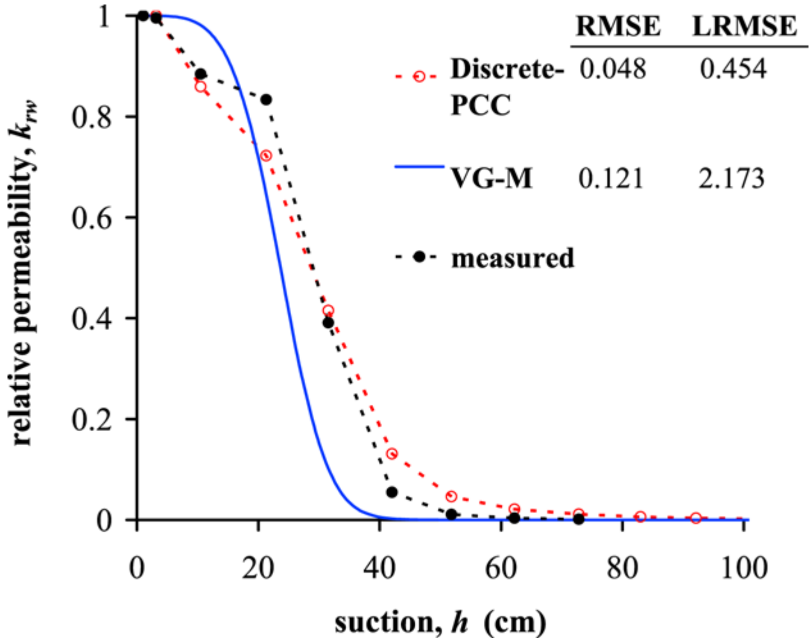

Figure 5. Discrete PCC and VG-M relative permeability predictions for Superstition sand.

where $r_{\max}$ is the largest pore size and $h_{\max}$ is the highest suction that drains the smallest pores present under the influence of capillary forces. Equation (17) was derived by assuming a continuous pore size distribution and has a form similar to the models of *Purcell* [1949] and *Burdine* [1953].

[14] In this study, we will concentrate on evaluating the discrete function given by equation (15) since its application does not require any fitting procedure and, unlike equation (17), it does not require a priori knowledge of the minimum suction ($h_{\min}$). We will test equation (15) against the commonly applied van Genuchten-Mualem (VG-M) [*van Genuchten*, 1980] approach using the UNSODA database [*Leij et al.*, 1996] and other previously published measurements of the water retention and relative permeability curves.

3. Data Sets and Model Testing

[15] UNSODA is one of the largest soil hydraulic data sets that includes suction–water content–relative permeability data for a wide range of soils from clay to gravel. Within UNSODA only 35 data sets are paired; that is, the water content and relative permeability measurements were collected at the same capillary pressures. We restricted our comparison to these paired data because we have not, as of yet, established a viable method to estimate the relative permeability at suctions between those included in the actual measurements. Another five paired data sets were located

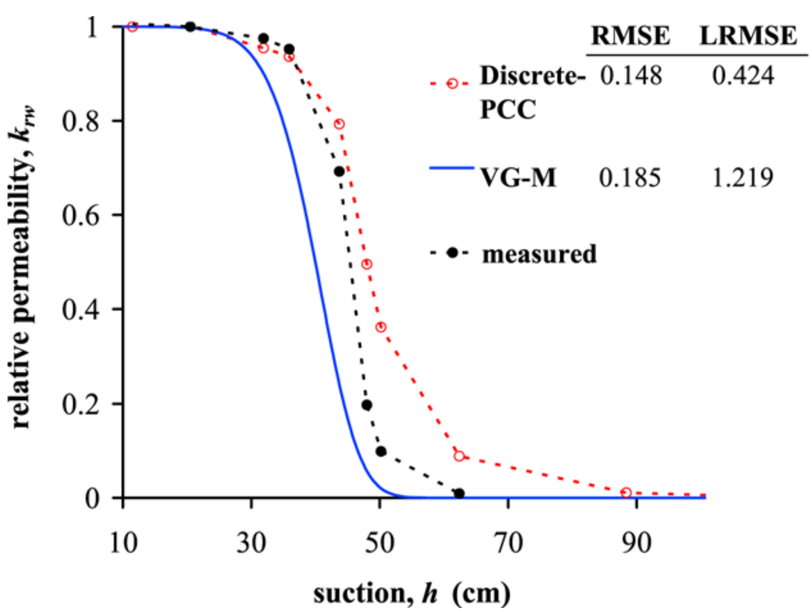

Figure 6. Discrete PCC and VG-M relative permeability predictions for Berea sandstone.

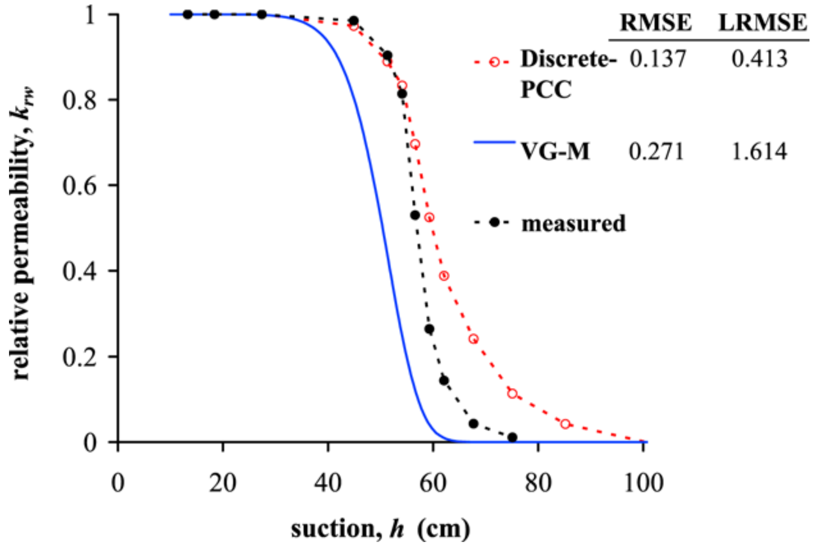

Figure 7. Discrete PCC and VG-M relative permeability predictions for Hygiene sandstone.

in the literature for more detailed analysis, including: Yolo light clay [*Moore*, 1939], Guelph loam [*Elrick and Bowman*, 1964], Superstition sand [*Richards*, 1953], Hygiene sandstone [*Brooks and Corey*, 1964], and Berea sandstone [*Brooks and Corey*, 1964]. The Yolo light clay and Guelph loam data sets also appear in the UNSODA database. These five data sets are well documented and have been extensively investigated [*van Genuchten*, 1980; *Fredlund et al.*, 1994].

[16] Predictions of equation (15) were compared with the popular empirical VG-M [*van Genuchten*, 1980] relative permeability function. The van Genuchten water retention and VG-M relative permeability are expressed as functions of suction in the following equations:

$$S = S_r + (1 - S_r)[1 + (\alpha h)^n]^{-m}; \qquad m = 1 - 1/n, \tag{18}$$

$$k_{rw}(h) = \frac{\left\{1 - (\alpha h)^{n-1}[1 + (\alpha h)^n]^{-m}\right\}^2}{[1 + (\alpha h)^n]^{m/2}}; \qquad m = 1 - 1/n, \tag{19}$$

where S is the saturation and S_r is the residual saturation. VG-M model parameters (α, n, and S_r) were obtained by fitting equation (18) to the measured water retention data sets using nonlinear regression (Marquardt method) as implemented by *SAS Institute* [1999] (SAS). All of the fits converged according to the SAS default convergence criterion [*SAS Institute*, 1999]. The average coefficient of determination (R^2) between the measured and predicted saturations for equation (18) fitted to the 40 water retention data sets was 0.999.

[17] The accuracy of the predictions of relative permeability by the two models was evaluated by the root-mean-square error (RMSE). We also computed the logarithmic RMSE (LRMSE) on the basis of the logarithms of the measured and predicted k_{rw} values to quantify the performance of both models at low relative permeabilities. Paired t tests were used to evaluate if the differences in RMSE or LRMSE values between the two models were statistically significant at $p < 0.05$ (5% level of significance).

4. Results

[18] Figure 2 shows the differences in the performance of the discrete PCC (red circles) and the VG-M (blue squares) models for the pooled 40 data sets. A 1:1 line shows the optimal performance. The discrete PCC predictions were generally closer to this line. The mean RMSE for the discrete PCC model was 0.128, while for the VG-M model, the mean RMSE was 0.140. The mean LRMSE for the discrete PCC model was 0.813, while for the VG-M model it was 1.555. Paired t tests for the 40 data sets showed that mean RMSEs of the two models were not significantly different, while the mean LRMSEs were significantly different at $p < 0.05$. The mean LMRSE may be a better measure of fit given the orders of magnitude range of relative permeability values. These results indicate that overall, the discrete PCC method (equation (15)) predicted the measured data better than the VG-M (equation (19)) at $p < 0.05$.

[19] We also present individual comparisons for the five data sets collected from the literature (Figures 3–7). The data sets presented within this manuscript include all that have

Table 1. Estimates of Model Parameters for the VG Model[a]

Soil	Reference	S_r	α (cm^{-1})	n	R^2
Yolo light clay	*Moore* [1939]	0.430	0.025	1.776	0.998
Guelph loam	*Elrick and Bowman* [1964] and *van Genuchten* [1980]	0.414	0.013	1.946	0.994
Superstition sand	*Richards* [1953]	0.287	0.028	5.100	0.999
Berea sandstone	*Brooks and Corey* [1964]	0.328	0.019	8.928	0.996
Hygiene sandstone	*Brooks and Corey* [1964] and *van Genuchten* [1980]	0.615	0.016	10.64	0.995

[a]Model parameters are α, n, and S_r. Fitted to measured water retention for five published paired data sets assuming $m = 1 - 1/n$.

been tested; no results were screened out. Table 1 presents the estimated parameters of the VG function fitted to the water retention data. For the discrete PCC model, the RMSEs ranged between 0.039 and 0.148 with a mean of 0.090. In contrast, the RMSEs from the VG-M model ranged between 0.113 and 0.271, with a mean of 0.179. The LRMSEs for the discrete PCC model ranged between 0.227 and 0.489 with a mean of 0.401, while the VG-M LRMSE values ranged between 0.820 and 2.173 with a mean of 1.501. The discrete PCC predictions resulted in smaller RMSE and LRMSE values in every case. Generally, the VG-M model underpredicted the relative permeabilities. Paired t test showed that the discrete PCC mean RMSE and mean LRMSE were significantly smaller than the VG-M model for these five soils ($p < 0.05$).

5. Discussion and Conclusions

[20] We combined the probability of drainage concept and the PCC approach introduced by *Cihan et al.* [2007] and *Cihan et al.* [2009], respectively, to derive the relative permeability function for monotonic drainage of random mass fractal porous media, i.e., the discrete PCC model. The discrete PCC model allows estimation of the relative permeability directly from measured water retention data and does not require curve fitting.

[21] The performance of the discrete PCC was tested on 40 data sets and compared with the VG-M model. Results indicate that overall, the discrete PCC method (equation (15)) predicted the relative permeability significantly better than the VG-M method. It should be noted that some data sets within the UNSODA database appear to be questionable. For instance, the relative permeability of some clay soils decreased rapidly with a small increase in suction. This might indicate the presence of macropores or fractures resulting in considerable momentum losses thereby flawing the assumptions behind Darcy's law and capillary equilibrium based on the Young-Laplace equation. In these cases, both models resulted in inadequate predictions of relative permeability.

[22] We also analyzed individual predictions of the models for five soils used by many researchers in previous publications on this subject. The VG-M model generally underpredicted the measured data for all of the five soils. In contrast, the discrete PCC model predicted the measured data reasonably well except for overpredicting k_{rw} at high suctions for the Berea and Hygiene sandstones (see Figures 6 and 7).

[23] The discrete PCC relative permeability function can be used within numerical algorithms to solve the partial differential equations governing unsaturated flow. However, some sort of interpolation scheme is needed to compute k_{rw} for suctions not included in the experimental water retention data set. The present model is restricted to monotonic drainage from saturation. There is no theoretical reason why it cannot be adapted to wetting and thereby extended to incorporate hysteresis. However, limited relative permeability data are available for model testing in the wetting case.

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